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THE DIFFRACTION OF SURFACE WAVES BY A FLOATING ELASTIC PLATE AT OBLIQUE INCIDENCE†

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The diffraction of plane surface waves by a floating semi-infinite elastic plate at oblique incidence in a fluid of finite depth is investigated. An exact solution of this problem is constructed by the Wiener–Hopf method. Analytic formulae for the velocity potential of the liquid and for the reflection and transmission coefficients are obtained. The behaviour of the plate on the waves, namely, the distribution of the displacements and deformations as a function of the dimensionless parameters of the problem, such as the angle of incidence and the reduced stiffness and depth, is investigated. © 2004 Elsevier Ltd. All rights reserved.

The behaviour of an elastic plate floating on waves has been studied previously as it applies to an ice sheet. Recently, interest in this problem was increased in connection with the design of artificial islands, floating airports and platforms of different kinds. A number of numerical methods exist for solving such problems (see the reviews [1, 2]). However, the applicability of numerical methods in the case of short waves is doubtful. Most of these methods require a refinement of the mesh for short waves, which leads to high-order matrix equations, and the convergence of the numerical algorithms has not been substantiated. The need therefore arises to develop methods which are also applicable for short waves. An analytic solution of this problem using the Wiener–Hopf method has been constructed [3–6] for a semi-infinite plate. The Wiener–Hopf technique allows of a variety of different approaches to the solution of the problem, and each publication uses its own approach.

A specific feature of the boundary-value problem which arises in hydroelasticity is the high order of the derivative in one of the boundary conditions. As a consequence of this, in the Wiener–Hopf method the solution depends on two constants, the determination of which in explicit form presents difficulties. Thus, in the case of oblique incidence a system of linear algebraic equations was obtained in [3] whose coefficients are a series in roots of the dispersion relation (a transcendental equation). The case of a stratified fluid was considered in [4] where the conditions at the free edge were not specified, and, without any justification, the constants were assumed to be zero. The case of normal incidence of a wave in a fluid of infinite depth was consideration [5], but the formulae obtained are so complicated that even the authors themselves do not attempt to use them to carry out calculations.

Numerical calculations using the formulae obtained can only be found in the paper by Balmforth and Craster [6] and only for the reflection and transmission coefficients. In that paper the elastic plate is described by the Timoshenko–Mindlin equation. The solution depends on two constants, but in addition to those two additional constants are introduced, and a system of four equations is obtained for them. Obviously the dimensionless variables were a poor choice, and as a result, the small term in the equation becomes equal to unity. A simple approximate formula for the reflection coefficient is derived, which is obtained on the assumption that both constants equal zero. The values of the reflection coefficient obtained using that formula are in good agreement with the corresponding values found taking into account the non-zero constants.

Another approach was proposed in [7, 8]. It has been shown that, in the case of normal incidence, if the small term is neglected, the system can be inverted, even without calculating its coefficients, and the approximate formula obtained in [6] is in fact an exact solution of the problem. Below we describe an approach that enables the constants to be determined and enables an exact solution to be obtained for the case of the oblique incidence of a wave on a semi-infinite plate floating on the surface of a fluid of finite depth.

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1. FORMULATION OF THE PROBLEM

The surface of an ideal incompressible fluid of finite depth H_0 is partially covered by a thin elastic semi-infinite plate. A plane wave of small amplitude is incident at an angle θ to the plane, where the wavelength is considerably greater than the thickness of the plate. We introduce a Cartesian system of coordinates (x, y, z) with the centre O on the edge of the plate, Ox axis directed perpendicular to the edge, Oy axis directed along the edge, and Oz axis directed vertically upwards. We shall neglect the settling of the plate into the fluid, and the boundary conditions will be referred to the undisturbed fluid surface. The problem is solved in the linear formulation.

The velocity potential of the liquid φ satisfies Laplace's equation

$$\Delta\varphi = 0 \tag{1.1}$$

The boundary conditions can be written in the form

$$z = -H_0 : \frac{\partial\varphi}{\partial z} = 0; \quad z = 0 : \frac{\partial\varphi}{\partial z} = \frac{\partial\eta}{\partial t} \tag{1.2}$$

$$z = 0 : D\Delta_h^2\eta + \rho_0 h\frac{\partial^2\eta}{\partial t^2} = p\ (x > 0),\ p = -\rho\left(\frac{\partial\varphi}{\partial t} + g\eta\right),\ \frac{\partial\varphi}{\partial t} + g\eta = 0\ \ (x < 0) \tag{1.3}$$

Here η is the vertical displacement of the upper surface of the fluid (the plate), g is the acceleration due to gravity, D is the cylindrical stiffness of the plate, h is its thickness, ρ and ρ_0 are the densities of the fluid and the plate and t is the time. The subscript h on the Laplace operator means that this operator is taken with respect to horizontal variables only. On the edge of the plate the moment and the intersecting force must vanish

$$z = 0, \quad x = 0 : \frac{\partial^2\eta}{\partial x^2} + \nu\frac{\partial^2\eta}{\partial y^2} = \frac{\partial}{\partial x}\left[\frac{\partial^2\eta}{\partial x^2} + (2-\nu)\frac{\partial^2\eta}{\partial y^2}\right] = 0 \tag{1.4}$$

where ν is Poisson's ratio.

The dependence of all the required functions on time and on the y coordinate is periodic and is expressed by the factor. $e^{i(ky - \omega t)}$ where k is the wave number with respect to the y coordinate of the incident wave, and ω is the frequency. We will introduce the dimensionless variables

$$\varphi' = \frac{\varphi}{A\sqrt{gl}}, \quad x' = \frac{x}{l}, \quad y' = \frac{y}{l}, \quad z' = \frac{z}{l}, \quad t' = \omega t, \quad H = \frac{H_0}{l}, \quad k' = kl$$

where A is the amplitude of the incident wave and $l = g/\omega^2$ is a characteristic length. Henceforth we shall omit the primes.

In terms of the dimensionless variables we represent the potential φ in the form

$$\varphi = \phi e^{i(ky-t)}, \quad \phi = \phi_0 + \phi_1, \quad \phi_0 = e^{i\gamma x}\frac{\operatorname{ch}(\mu(z+H))}{\operatorname{ch}(\mu H)}; \quad \gamma = \mu\cos\theta, \quad k = \mu\sin\theta$$

where ϕ_0 is the potential of the incident wave, ϕ_1 is the diffracted potential, γ and k are the wave-numbers of the incident wave corresponding to the x and y coordinates, and μ is found from the dispersion relation for waves on the surface of a fluid of depth. H, namely, μ th $(\mu H) - 1 = 0$.

Then from problem (1.1)–(1.4) one can obtain the boundary-value problem for ϕ_1

$$\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial z^2} - k^2\phi_1 = 0, \quad -H < z < 0 \tag{1.5}$$

$$z = -H : \frac{\partial\phi_1}{\partial z} = 0 \tag{1.6}$$

$$z = 0 : \frac{\partial\phi_1}{\partial z} - \phi_1 = 0(x < 0), \quad \left[\beta\left(\frac{\partial^2}{\partial x^2} - k^2\right)^2 + 1 - \delta\right]\frac{\partial\phi_1}{\partial z} - \phi_1 = Be^{i\gamma x}(x > 0) \tag{1.7}$$

$$z = 0, \quad x = 0: \left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\frac{\partial\phi}{\partial z} = \frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial x^2} + (2-\nu)\frac{\partial^2}{\partial y^2}\right]\frac{\partial\phi}{\partial z} = 0 \tag{1.8}$$

$$\beta = \frac{D}{\rho g l^4}, \quad \delta = \frac{\rho_0 h}{\rho l}, \quad B = \delta - \beta(\gamma^2 + k^2)^2$$

Here β, δ, θ and H are the dimensionless parameters of the problem.

In addition, the radiation conditions as $|x| \to \infty$ and the conditions of the regularity near the front edge (local boundedness of the energy) must be satisfied. In view of the assumptions made, the parameter $\delta \ll 1$, and hence in what follows we put $\delta = 0$; this approximation was also used in a number of other papers ([5], etc.).

2. THE DISPERSION RELATIONS

We will consider how the waves propagate in a fluid with a free surface and under the plate. We shall seek a solution of Eq. (1.5) of the form

$$e^{i\alpha x}\Psi(\alpha, z); \quad \Psi(\alpha, z) = \operatorname{ch}(\sqrt{\alpha^2 + k^2}(z + H))/\operatorname{ch}(\sqrt{\alpha^2 + k^2}H) \tag{1.9}$$

with condition (1.6) and the corresponding condition (1.7) on the upper boundary.

Surface waves. For surface waves the values of α must satisfy the dispersion relation

$$\mu\,\operatorname{th}(\mu H) - 1 = 0, \quad \mu = \sqrt{\alpha^2 + k^2}$$

This equation has two real roots $\pm\mu_0$ and a denumerable set of pure imaginary roots $\pm\mu_j$ ($j = 1, 2, \ldots$), situated symmetrically about real axis. Two real values $\pm\gamma(\gamma = \sqrt{\mu_0^2 - k^2})$ and pure imaginary roots $\pm\gamma_j(j = 1, 2, \ldots; \gamma_j = \sqrt{\mu_j^2 - k^2})$ correspond to these.

Flexural-gravitational waves. For waves propagating in the plate (so called flexural-gravitational waves) the dispersion relation has the form

$$\mu\,\operatorname{th}(\mu H) - 1 = 0, \quad \mu = \sqrt{\alpha^2 + k^2}$$

This equation has two real roots $\pm\lambda_0$, a denumerable set of pure imaginary roots $\pm\lambda_j$ ($j = 1, 2, \ldots$), symmetrical about the real axis and, in addition, four complex roots, symmetrical about the real and imaginary axes: λ_{-1} is the root located in the first quadrant and λ_{-2} is the root located in the second quadrant. The values $\pm\alpha_j$ ($j = -2, -1, 0, 1, \ldots; \alpha_j \sqrt{\mu_j^2 - k^2}$) correspond to these roots of the dispersion relation. If $\lambda_0 > k$, α_0 has a real value, otherwise it is imaginary.

Real roots of the dispersion relations specify propagating waves, whereas all the remaining roots specify edge waves, that decay exponentially far from the edge of the plate. The critical angle of incidence corresponds to $\lambda_0 = k$ and is defined by the formula $\theta_* = \arcsin(\lambda_0/\mu_0)$. If the angle of incidence is greater than the critical angle in the plate only edge waves exist.

3. ANALYTICAL SOLUTION OF THE PROBLEM

The solution of the problem will be constructed by the Wiener–Hopf method in the Jones interpretation [9]. We introduce into consideration the functions of the complex variable α

$$\begin{gathered}\Phi_+(\alpha, z) = \int_0^\infty e^{i\alpha x}\phi_1(x, z)dx, \quad \Phi_-(\alpha, z) = \int_{-\infty}^0 e^{i\alpha x}\phi_1(x, z)dx \\ \Phi(\alpha, z) = \Phi_+(\alpha, z) + \Phi_-(\alpha, z)\end{gathered} \tag{3.1}$$

The function $\Phi_+(\alpha, z)$ is defined in the upper half-plane $\{\operatorname{Im}\alpha > 0\}$, and $\Phi_-(\alpha, z)$ in the lower half-plane $\{\operatorname{Im}\alpha < 0\}$. By means of analytic continuation they can be defined in the whole complex plane.

We will investigate the behaviour of the functions $\Phi_\pm(\alpha, z)$ when $x \to -\infty$ the diffracted potential is a reflected wave of the form $\mathrm{R}e^{-i\gamma x}$ and a set of exponentially decaying waves. The least decaying wave corresponds to the root γ_1. Hence the function $\Phi_-(\alpha, z)$ is analytic in the half-plane $\{\mathrm{Im}\alpha < |\gamma_1|\}$ with the exception a pole at $\alpha = \gamma$. When $x \to \infty$, the potential φ_1 corresponds to the sum of waves, namely, a transmitted wave with wave number α_0, a wave with wave-number γ, which compensates φ_0, and a set of exponentially decaying modes. Consequently, the function $\Phi_+(\alpha, z)$ is analytic in the half-plane $\{\mathrm{Im}\alpha > c\}$ with the exception of poles $\alpha = -\alpha_0$ and $\alpha = -\gamma$; the quantity c equals the least imaginary part of the wave numbers of decaying modes in the plate.

The function $\Phi(\alpha, z)$ is the Fourier transform of the function $\varphi_1(x, z)$ and satisfies the equation

$$\frac{\partial^2\Phi}{\partial z^2} - (\alpha^2 + k^2)\Phi = 0$$

The general solution of this equation with condition (1.6) on the bottom has the form

$$\Phi(\alpha, z) = C(\alpha)\Psi(\alpha, z) \tag{3.2}$$

The function $\Psi(\alpha, z)$ is defined by expression (1.9).

We denote by $D_\pm(\alpha)$ expressions of the type (3.1), where the function φ_1 is replace by the left-hand side of the first boundary condition of (1.7), and we denote by $F_\pm(\alpha)$ similar expressions, in which the left-hand side of the second condition of (1.7) is taken as the integrand. We shall regard these integrals as the Fourier transforms of generalized functions [10]. For these functions the following relations are satisfied

$$D_+(\alpha) + D_-(\alpha) = C(\alpha)K_1(\alpha), \quad F_+(\alpha) + F_-(\alpha) = C(\alpha)K_2(\alpha) \tag{3.3}$$

where $K_1(\alpha)$ and $K_2(\alpha)$ are the dispersion functions for waves on the free surface and under the plate respectively

$$K_1(\alpha) = \sqrt{\alpha^2 + k^2}\,\mathrm{th}(\sqrt{\alpha^2 + k^2}H) - 1$$

$$K_2(\alpha) = (\beta(\alpha^2 + k^2)^2 + 1)\sqrt{\alpha^2 + k^2}\,\mathrm{th}(\sqrt{\alpha^2 + k^2}H) - 1$$

From boundary conditions (1.7) we have

$$D_-(\alpha) = 0, \quad F_+(\alpha) = -\frac{B}{i(\alpha + \gamma)}$$

Taking this into account, from relations (3.3) we find

$$D_+(\alpha) = C(\alpha)K_1(\alpha), \quad F_-(\alpha) - \frac{B}{i(\alpha + \gamma)} = C(\alpha)K_2(\alpha) \tag{3.4}$$

From the last two equations we obtain

$$D_+(\alpha) = K(\alpha)\left(F_-(\alpha) - \frac{B}{i(\alpha + \gamma)}\right), \quad K(\alpha) = \frac{K_1(\alpha)}{K_2(\alpha)} \tag{3.5}$$

According to the Wiener–Hopf method, it is necessary to factorize the function $K(\alpha)$, that is, to represent it in the form

$$K(\alpha) = K_+(\alpha)K_-(\alpha) \tag{3.6}$$

where the functions $K_\pm(\alpha)$ are regular in the same domains where the functions $\Phi_\pm(\alpha, z)$ are regular, respectively. The function $K(\alpha)$ has zeros and poles at the points $\pm\gamma$ and $\pm\alpha_0$. Therefore, we shall consider the domains of analyticity of S_+ and S_-, where S_+ is the half-plane $\{\mathrm{Im}\alpha < c\}$ with cuts, excluding the points $-\alpha_0$ and $-\gamma$, and S_- is the half-plane $\{\mathrm{Im}\alpha > -c\}$ with cuts, excluding the points α_0 and γ.

We introduce the function

$$g(\alpha) = \frac{K(\alpha)\beta(\alpha^2 - \alpha_0^2)(\alpha^2 - \alpha_{-1}^2)(\alpha^2 - \alpha_{-2}^2)}{\alpha^2 - \gamma^2}$$

This function has no zeros, is bounded and tends to unity at infinity. We factorize $g(\alpha)$ as follows [9]:

$$g(\alpha) = g_+(\alpha)g_-(\alpha), \quad g_\pm(\alpha) = \exp\left[\pm\frac{1}{2\pi i}\int_{-\infty\mp ic}^{\infty\mp ic}\frac{\ln g(x)}{x-\alpha}dx\right] \tag{3.7}$$

We define the function $K_\pm(\alpha)$ by the expressions

$$K_\pm(\alpha) = \frac{(\alpha\pm\gamma)g_\pm(\alpha)}{\sqrt{\beta}(\alpha\pm\alpha_0)(\alpha\pm\alpha_{-1})(\alpha\pm\alpha_{-2})}$$

Here $K_+(\alpha) = K_-(-\alpha)$.

We now write Eq. (3.5) in the form

$$K_-(\alpha)F_-(\alpha) - \frac{B}{i(\alpha+\gamma)}(K_-(\alpha) - K_-(-\gamma)) = \frac{D_+(\alpha)}{K_+(\alpha)} + \frac{BK_-(-\gamma)}{i(\alpha+\gamma)}$$

On the left-hand side we have a function that is analytic in the domain S_-, and on the right-hand side we have a function that is analytic in S_+. By analytic continuation this function can be defined over the whole complex plane, and by Liouville's theorem, this function is a polynomial. The degree of the polynomial is determined from the behaviour of the function as $|\alpha| \to \infty$.

From the condition for the energy to be locally bounded it follows that near the edge of the plate the velocities have a singularity of order not higher than $O(r^{-\lambda})$, where $\lambda < 1$ and r is the distance to the edge of the plate. Then as $|\alpha| \to \infty$ the function $F_-(\alpha)$ has an order not higher than $O(|\alpha|^{\lambda+3})$ [10], whereas the function $D_+(\alpha)$ has an order not higher than $O(|\alpha|^{\lambda-1})$. The functions $K_\pm(\alpha)$ are of order $O(|\alpha|^{-2})$ at infinity, since $g_\pm(\alpha) \to 1$ as $|\alpha| \to \infty$. Consequently, the degree of the polynomial equals unity. We obtain

$$\frac{D_+(\alpha)}{K_+(\alpha)} + \frac{BK_-(-\gamma)}{i(\alpha+\gamma)} = \frac{BK_-(-\gamma)}{i}(a + b\alpha)$$

where a and b are unknown constants, which are found from conditions (1.8).

Expressing $D_+(\alpha)$ from the last equation and taking into account Eqs (3.2) and (3.4), we find

$$\phi_1(x, z) = \frac{BK_-(-\gamma)}{2\pi i}\int_{-\infty}^{\infty} e^{-i\alpha x}\Psi(a, z)\frac{K_+(\alpha)}{K_1(\alpha)}\left(a + b\alpha - \frac{1}{\alpha+\gamma}\right)d\alpha \tag{3.8}$$

The contour of integration has to be chosen in such a way that it lies completely at the intersection of the domains S_+ and S_-. One can choose the contour of integration on the real axis, going around the points α_0 and γ from below and around the points $-\alpha_0$ and γ from above.

When $x > 0$ we multiply and divide the integrand by $K_-(\alpha)$ and close the contour of integration in the lower half-plane. We obtain poles at the points $-\gamma$, $-\alpha_j$ ($j = -2, -1, \ldots$). We evaluate the integral using the theory of residues. We obtain

$$\frac{\partial\phi}{\partial z}(x, 0) = -BK_-(-\gamma)\sum_{j=-2}^{\infty} e^{i\alpha_j x}\frac{\sqrt{\alpha_j^2 + k^2}\,\mathrm{th}(\sqrt{\alpha_j^2 + k^2}H)}{K_-(-\alpha_j)K_2'(-\alpha_j)}\left(\alpha - b\alpha_j - \frac{1}{\gamma - \alpha_j}\right) \tag{3.9}$$

From the dispersion relation under the plate we have

$$\sqrt{\alpha_j^2 + k^2}\,\mathrm{th}(\sqrt{\alpha_j^2 + k^2}H) = -\frac{K_1(-\alpha_j)}{\beta(\alpha_j^2 + k^2)^2}$$

Substituting this expression into equality (3.9) and the result obtained into boundary conditions (1.8), we arrive at a system of linear second-order algebraic equations for finding the unknown constants a and b

$$\begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{Vmatrix} \begin{Vmatrix} a \\ b \end{Vmatrix} = \begin{Vmatrix} C_1 \\ C_2 \end{Vmatrix}$$

where

$$A_{11} = U_2 - (1-\nu)k^2 U_0, \quad A_{12} = -U_3 + (1-\nu)k^2 U_1, \quad A_{21} = U_3 + (1-\nu)U_1$$

$$A_{22} = -U_4 + \nu k^2 U_2 - (1-\nu)k^4 U_0, \quad C_1 = \tilde{U}_2 - (1-\nu)k^2 \tilde{U}_0, \quad C_2 = \tilde{U}_3 + (1-\nu)\tilde{U}_1$$

$$U_m = \sum_{j=-2}^{\infty} U_{mj}, \quad \tilde{U}_m = \sum_{j=-2}^{\infty} \frac{U_{mj}}{\gamma - \alpha_j}, \quad U_{mj} = \frac{K_1(\alpha_j)\alpha_j^{m-2[m/2]}}{(\alpha_j^2 + k^2)^{2-[m/2]} K_-(-\alpha_j) K_2'(-\alpha_j)}$$

($[m/2]$ is the integer part of the quantity $m/2$).

We will now calculate U_m and $\tilde{U}_m$. To do this, we replace the sums by integrals using the theory of residues, multiply and divide the integrand by $K_+(\alpha)$, and close the contour of integration in the upper half-plane. We obtain

$$U_4 = 0, \quad U_m = V_m(ik) + V_m(-ik), \quad \tilde{U}_m = \tilde{V}_m(ik) + \tilde{V}_m(-ik)$$

$$V_0(\pm ik) = \frac{1}{4k^2\beta}\left[\frac{K_+(\pm ik)}{\pm ik} - K_+'(\pm ik)\right], \quad V_1(\pm ik) = -\frac{K_+'(\pm ik)}{\pm 4ik\beta}$$

$$V_2(\pm ik) = \frac{K_+(\pm ik)}{\pm 2ik\beta}, \quad V_3(\pm ik) = -\frac{K_+(\pm ik)}{2\beta}$$

$$\tilde{V}_0(\pm ik) = \frac{1}{4k^2\beta}\left[\frac{K_+(\pm ik)}{(\gamma \pm ik)^2} + \frac{K_+(\pm ik)}{\pm ik(\gamma \pm ik)} - \frac{K_+'(\pm ik)}{\gamma \pm ik}\right]$$

$$\tilde{V}_1(\pm ik) = \frac{1}{\pm 4ik\beta(\gamma \pm ik)}\left[\frac{K_+(\pm ik)}{\gamma \pm ik} - K_+'(\pm ik)\right]$$

$$\tilde{V}_2(\pm ik) = \frac{K_+(\pm ik)}{\pm 2ik\beta(\gamma \pm ik)}, \quad \tilde{V}_3(\pm ik) = -\frac{K_+(\pm ik)}{2\beta(\gamma \pm ik)}$$

The case of normal incidence of the wave, the values of the constants [7] $a = 1/\gamma$ and $b = -1/\gamma^2$ are obtained. In the general case we represent the desired constants in the form

$$a = \frac{\gamma}{\gamma^2 + k^2} + a_1, \quad b = -\frac{1}{\gamma^2 + k^2} + b_1$$

Solving the system, we obtain

$$a_1 = \frac{2(1-\nu)ik\Delta_1}{(\gamma^2 + k^2)\Delta}, \quad b_1 = \frac{2(1-\nu)ik\Delta_2}{(\gamma^2 + k^2)\Delta}$$

$$\Delta_1 = -(3+\nu)ikD_2^+ + \frac{2(1-\nu)ik(2k^2 D_1 - \gamma)}{\gamma^2 + k^2}$$

$$\Delta_2 = (3+\nu)D_2^- + \frac{2(1-\nu)ik(2\gamma D_1 - 1)}{\gamma^2 + k^2}$$

$$\Delta = (3+\nu)(1-\nu)[K_+^2(ik)+K_+^2(-ik)] + 2(5+2\nu+\nu^2) - 4(1-\nu)^2 k^2 D_1^2$$

$$D_2^{\pm} = \frac{K_+^2(ik)}{\gamma+ik} \pm \frac{K_+^2(-ik)}{\gamma-ik}, \quad D_1 = \frac{K_+'(ik)}{K_+(ik)} = \frac{K_+'(-ik)}{K_+(-ik)}$$

The last relation follows from relation (3.6) and the fact that $K_+(-\alpha) = K_-(\alpha)$.

Substituting the values of coefficients a and b into expression (3.8) and taking into account relations (3.6), we obtain the formula for the potential

$$\phi_1(x,z) = \frac{\beta(\gamma^2+k^2)K_+(\gamma)}{2\pi i} \times$$

$$\times \int_{-\infty}^{\infty} e^{-i\alpha x}\Psi(\alpha,z)\frac{K_+(\alpha)}{K_1(\alpha)}\left[\frac{\alpha^2+k^2}{\gamma+\alpha} - \frac{2(1-\nu)ik}{\Delta}S(\alpha)\right]d\alpha \tag{3.10}$$

$$S(\alpha) = (3+\nu)\frac{\alpha-ik}{\gamma+ik}K_+^2(ik) - (3+\nu)\frac{\alpha+ik}{\gamma-ik}K_+^2(-ik) +$$

$$+\frac{2(1-\nu)ik}{\gamma^2+k^2}[2(\gamma\alpha+k^2)D_1 - (\gamma+\alpha)]$$

4. THE TRANSMISSION AND REFLECTION COEFFICIENTS, THE ELEVATION OF THE FREE BOUNDARY, AND THE DEFLECTION AND DEFORMATIONS OF THE PLATE

We can now obtain the reflected and transmitted waves. When $|x| \to \infty$, the potential has the form

$$\phi(x,0) = e^{i\gamma x} + Re^{-i\gamma x}, \quad x \to -\infty, \quad \phi(x,0) = Te^{i\alpha_0 x}, \quad x \to \infty$$

If the angle of incidence of the wave is greater than the critical angle, the quantity T defines the complex amplitude of the least decaying wave. Taking the residue at the point $\alpha = \gamma$ in expression (3.10), we obtain

$$R = \frac{\beta(\gamma^2+k^2)K_+^2(\gamma)}{K_1'(\gamma)}\left[\frac{\gamma^2+k^2}{2\gamma} - \frac{2(1-\nu)ik}{\Delta}S(\gamma)\right]$$

The quantity T is defined by the residue at the point $\alpha = -\alpha_0$

$$T = \frac{\beta(\gamma^2+k^2)K_+(\gamma)}{K_+(\alpha_0)K_2'(\alpha_0)}\left[\frac{\alpha_0^2+k^2}{\gamma-\alpha_0} - \frac{2(1-\nu)ik}{\Delta}S(-\alpha_0)\right]$$

In the case under consideration the amplitude $|R|$ is the reflection coefficient, in view of the normalization performed. We determine the transmission coefficient by the value of $|T|$. We will find an expression for $|K_+(\gamma)|$. We deform the contour of integration in the formula (3.7) into the real axis. Using Sokhotsky's theorem we find

$$|K_+(\gamma)| = \sqrt{\frac{2\gamma|\gamma-\alpha_0|K_1'(\gamma)}{|\gamma+\alpha_0|K_2(\gamma)}}$$

For the reflection coefficient we obtain

$$|R| = \frac{|\gamma-\alpha_0|}{|\gamma+\alpha_0|}\left|1 - \frac{4(1-\nu)ik\gamma}{(\gamma^2+k^2)\Delta}S(\gamma)\right|$$

If the angle of incidence of the wave is less than the critical angle, α_0 is a real quantity and

$$|K_+(\alpha_0)| = \sqrt{\frac{(\gamma+\alpha_0)|K_1(\alpha_0)|}{2\alpha_0(\gamma-\alpha_0)K_2'(\alpha_0)}}$$

Then the formula for $|T|$ takes the form

$$|T| = \frac{2}{\gamma+\alpha_0}\sqrt{\frac{\gamma\alpha_0 K_1'(\gamma)[\beta(\alpha_0^2+k^2)^2+1]}{K_2'(\alpha_0)}}\left|1-\frac{2(1-\nu)ik(\gamma-\alpha_0)}{(\alpha_0^2+k^2)\Delta}S(-\alpha_0)\right|$$

In the case when the angle of incidence of the wave is greater than the critical angle, α_0 is a pure imaginary quantity and $|\gamma-\alpha_0| = |\gamma+\alpha_0|$. We obtain

$$|T| = \frac{\sqrt{2\gamma\beta K_1'(\gamma)}}{|K_2'(\alpha_0)K_+(\alpha_0)|}\left|\frac{\alpha_0^2+k^2}{\gamma-\alpha_0}-\frac{2(1-\nu)ik}{\Delta}S(-\alpha_0)\right|$$

There is an exact energy relation [6] between the amplitudes $|R|$ and $|T|$

$$|R|^2+|T|^2\mathrm{Re}\left(\frac{K_2'(\alpha_0)}{K_1'(\gamma)[\beta(\alpha_0^2+k^2)^2+1]}\right) = 1 \qquad (4.1)$$

Calculations show that the expressions obtained for the amplitudes, obey this relation exactly. From (4.1) it follows that in the case when the angle of incidence of the wave is greater than the critical angle, $|R| = 1$.

The vertical displacements of the plate and the free surface are found from the relation $\eta = i\phi_z(x, 0)$. For the elevation of the free surface we obtain the formula

$$\eta_-(x) = ie^{i\gamma x}+iRe^{-i\gamma x}+i\beta(\gamma^2+k^2)K_+(\gamma)\sum_{j=1}^{\infty}e^{i\gamma_j x}\frac{K_+(\gamma_j)}{K_1'(\gamma_j)}\left[\frac{\gamma_j^2+k^2}{\gamma+\gamma_j}-\frac{2(1-\nu)ik}{\Delta}S(\gamma_j)\right]$$

The first term is the incident wave, the second term is the reflected wave and the remaining terms represent edge effects (modes which decay exponentially far from the edge).

For the deflection of the plate we obtain

$$\eta_+(x) = i\beta(\gamma^2+k^2)K_+(\gamma)\sum_{j=-2}^{\infty}e^{i\alpha_j x}\frac{K_1(\alpha_j)+1}{K_+(\alpha_j)K_2'(\alpha_j)}\left[\frac{\alpha_j^2+k^2}{\gamma-\alpha_j}-\frac{2(1-\nu)ik}{\Delta}S(-\alpha_j)\right]$$

The nature of the distribution of the deflection amplitude of the plate depends on the angle of incidence of the wave on the plate. If the angle of incidence of the wave is less than the critical angle, as $x \to \infty$ the plate displacement is a travelling wave with amplitude

$$|\eta| = |T|\lambda_0 \mathrm{th}(\lambda_0 H), \quad \lambda_0 = \sqrt{\alpha_0^2+k^2}$$

If the angle of incidence of the wave is greater than the critical angle, all the roots $\alpha_j (j = -2, -1, 0, \ldots)$ have an imaginary part, and the plate displacements decay far from the edge. When the angle of incidence is close to the critical value and slightly exceeds it, the quantity α_0 has a pure imaginary value, close to zero. Then the decay of the deflection amplitude is very slow. Our calculations have shown that the maximum deflection amplitudes are observed at the edge. The effect of the depth of the fluid on the amplitude of the plate bending turns out to be weak.

The strain tensor of the plate in dimensional variables has the form

$$e = -\frac{hA}{2l^2}\left\|\begin{matrix}\dfrac{\partial^2\eta}{\partial x^2} & \dfrac{\partial^2\eta}{\partial x\partial y}\\[2mm] \dfrac{\partial^2\eta}{\partial x\partial y} & \dfrac{\partial^2\eta}{\partial y^2}\end{matrix}\right\|$$

At every point there is a direction along which maximum deformation is reached. The maximum deformation of the plate $e_{\max}$ is calculated as the eigenvalue of the matrix of greatest absolute magnitude $e_{\max}(x) = hAe_m(x)/2l^2)$, where $e_m(x)$ is the dimensionless maximum deformation. For normal incidence of the wave the maximum deformation is identical with the normal deformation $e_n = e_{xx}$ and they both vanish at the edge. In the case of oblique incidence the deformations at the edge are non-zero.

5. NUMERICAL RESULTS

Numerical calculations for a semi-infinite sheet of ice of thickness $h = 1.5$ m were carried out for the following values of the parameters [6]: Young's modulus $E = 6 \times 10^9$ Pa, density of ice 922.5 kg/m^3, and density of water 1025 kg/m^3.

In Fig. 1 we compare the relations obtained for the reflection and transmission coefficients as a function of the frequency for this plate for an angle of incidence of the wave $\theta = \pi/3$ (the solid curves) and the corresponding curves obtained in [6], taking into account the parameter δ (the dot-dash curves). The dimensionless depth of the fluid was taken to be equal to 100, whereas in [6] this quantity was equal to infinity. A difference in the curves is only detectable near the critical values of the parameters. It should be noted that the approach used by Balmforth and Claster leads to a system of four equations, which must be satisfied at the points $\pm\sqrt{\pm\sqrt{\delta/\beta} - k^2}$. If δ is small and β is large, the system is close to degenerate.

A numerical investigator of the dependence of the reflection and transmission coefficients on the dimensionless parameters, namely, the angle of incidence of the wave, the dimensionless stiffness of the β and the fluid depth H, was carried out. In Fig. 2 we show the reflection and transmission coefficients as a function of the parameter β for $\theta = \pi/6$ and $\theta = \pi/3$. The solid curves correspond to $H = 100$ and the dashed curves to $H = 1$. The results for the case of normal incidence were presented before in [7]. It follows from the graphs that the effect of the depth is insignificant, whereas the influence of the

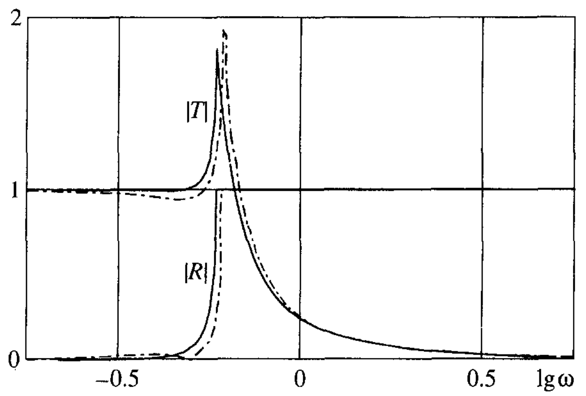

Fig. 1

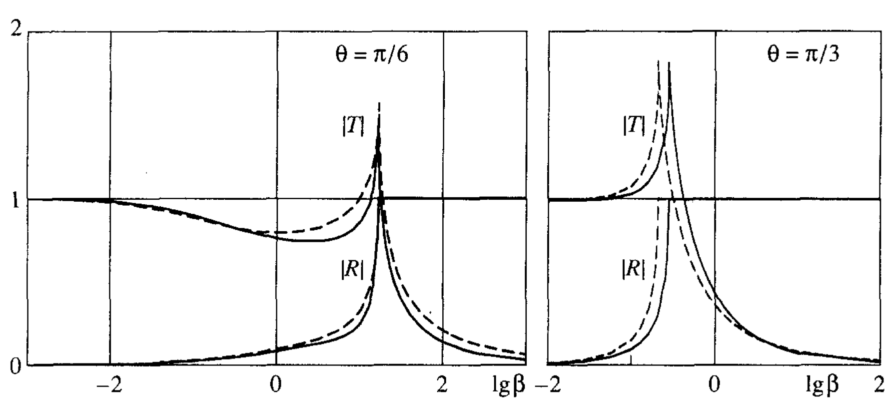

Fig. 2

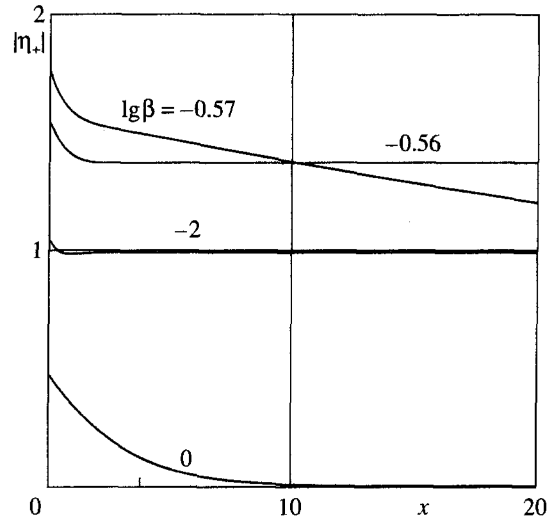

Fig. 3

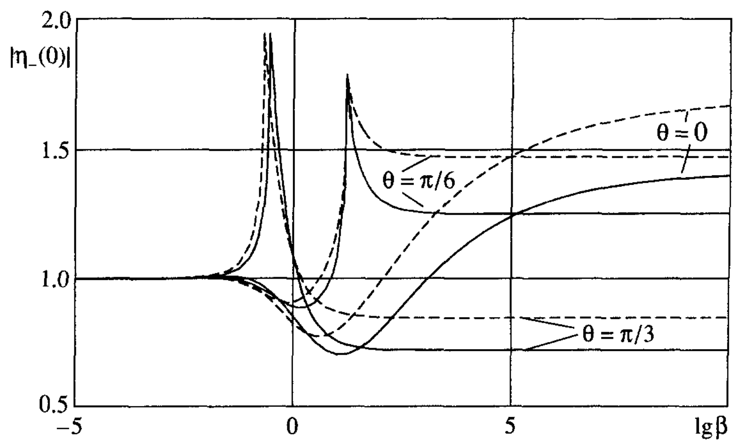

Fig. 4

parameters β and θ is strong. The maximum values of the transmission coefficients are observed at critical values of the parameters.

In Fig. 3 we show that distribution of the amplitude of the deflection of the sheet for various values of the parameter β, for an angle of incidence $\theta = \pi/3$ and a dimensionless depth $H = 100$. The critical value of the parameter β is equal to –0.56063. The value $\lg\beta = -0.56$ is subcritical, and $\lg\beta = -0.57$ is supercritical.

The dependence of the amplitude of the elevation of the wave at the edge on the parameter β for different angles of incidence of the wave is shown in Fig. 4; the solid curves correspond to a depth $H = 100$ and the dashed curves correspond to a depth $H = 1$. As can be seen from the curves, the effect of the depth is significant only at a large and supercritical values of the parameter β. For oblique incidence of the wave the maximum amplitudes of elevation of the wave at the edge occur at the critical value of the angle. It follows from the calculations that the amplitude of elevation of the free surface is a maximum at the edge. The effect of the depth of the liquid on the elevation of the free surface turned out to be stronger than on the displacement of the sheet.

In Fig. 5(a) we show the dependence of the amplitude of the deflection of the sheet at the edge for $H = 100$ and various values of the angle of incidence of the wave. The critical values of the parameter β are as follows: $\beta_* = -0.56063$ for $\theta = \pi/3$ and $\beta_* = 1.2104$ for $\theta = \pi/6$. For $\theta = 0$ all values of the parameter β are subcritical. At the critical value of the angle θ the amplitude reaches a maximum; this maximum can be both local and global.

The amplitudes of the displacements of the sheet and of the elevation of the free boundary at the edge are different. In Fig. 5(b) we show the dependence of the level difference at the edge on the

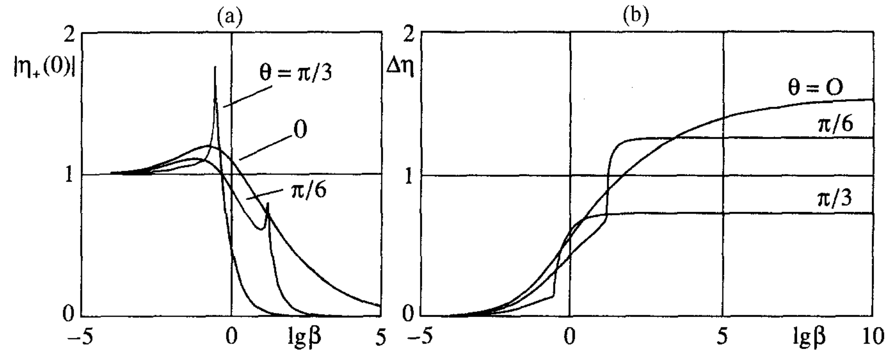

Fig. 5

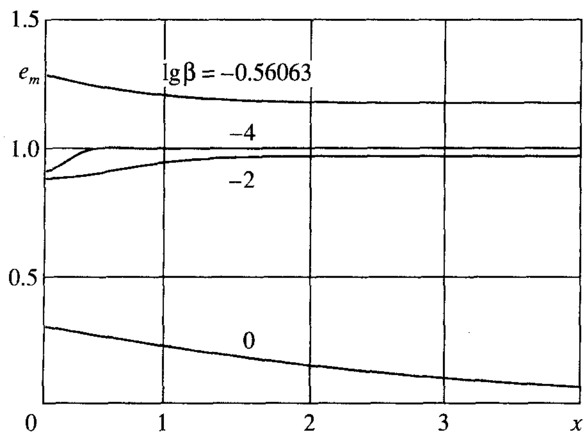

Fig. 6

parameter β for various angles of incidence of the wave for $H = 100$. As can be seen from the graph, the maximum amplitudes of the level difference are reached for normal incidence of the wave and large values of the parameter β, that is, for short waves. For oblique incidence of the waves and critical values of the parameters (corresponding to the critical angle) a sharp increase in the amplitude of the level difference is observed. If the thickness of the sheet is greater than the amplitude of the level difference, the edge of the sheet will periodically emerge from the water and then flop onto it. In that case it is necessary to use another model that takes the impact into account.

Our calculations show that the deformations of the sheet depend strongly on the parameter β and the angle of incidence of the wave and depend only slightly on the depth of the fluid. If the angle of incidence of the wave is greater than critical angle, far from the edge there are no deformations of the sheet, since in this case all modes decay. Hence the largest value of $e_m(x)$ is reached at the edge. The least decaying mode is one corresponding to the root α_0. If the angle of incidence of the wave is subcritical, the largest values of $e_m(x)$ can be reached both at the edge and at some distance from it, in particular, at infinity. In Fig. 6 we show the distribution of the dimensionless maximum deformations $e_m(x)$ over the sheet for various values of the parameter β for an angle of incidence of the wave $\theta = \pi/3$ and $H = 100$; the values $\lg\beta = -4$ and -2 correspond to the subcritical case, the value $\lg\beta = -0.56063$ is close to the critical case, and $\lg\beta = 0$ corresponds to the supercritical case.

At small values of the parameter β (that is, for very long incident waves) the largest value of $e_m(x)$ is reached at a certain finite distance from the edge (in Fig. 6 this is the curve corresponding to $\beta = 10^{-4}$). As the parameter β increases, the largest value of $e_m(x)$ is reached at infinity (the curve corresponding to $\beta = 10^{-2}$. When β is increased further, the largest value of $e_m(x)$ is shifted to the leading edge (the curve corresponding to $\lg\beta = -0.56063$). At supercritical values of the parameter β the function $e_m(x)$ decays exponentially far from the edge (the curves corresponding to $\beta = 1$).

If the value of the parameter β is fixed, then when the angle of incidence of the wave increases the maximum deformations at the edge increase, so long as the angle of incidence of the wave is less than the critical value, and decrease if the angle becomes greater than the critical value.

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